Overview

1. Motivation for BMO space
2. BMO Space: An Introduction
3. BMO and $L^\infty$: John-Nirenburg inequality
4. BMO and $L^p$
5. Applications of J-N’s inequality
6. BMO-$H^1$ Duality
Motivation for BMO

- Fritz John’s work on (nonlinear) elasticity, based on Agmon-Douglis-Nirenberg’s previous work.
- Deformation of an elastic body $\Omega \subset \mathbb{R}^3$ is a function $f : \Omega \mapsto \mathbb{R}^3$.
- Elasticity equation: $Df(x) = R(x)E(x)$, where $R(x)$ is a rotation and $E(x) = [(Df(x))^T Df(x)]^{1/2}$.
- The nonlinear elastic energy controls the nonlinear strain $|E(x) - I|$ but not the local rotation $R(x)$.
- Linear elasticity: Korn’s inequality gives $L^2$ estimate of the infinitesimal rotation in terms of $L^2$ norm of the linear stain.
- John’s result: extend it to nonlinear case, that is, if the nonlinear strain is uniformly small on a cube then the BMO norm (as we can call after John-Nirenberg define it) of $Df$ is also small.
- $\|E(x) - I\|_{L^\infty(Q)} \leq \epsilon$ implies that $\|Df\|_{\text{BMO}(Q)} \leq C\epsilon$ for sufficiently small $\epsilon$.
- The estimate is actually an estimate on the oscillation of $R(x)$ knowing that $E$ stays close to $I$ by hypothesis.
Motivation for BMO

- Knowing Nirenbergs analytical power and his love of inequalities, John drew Nirenberg into exploring the implications of his result on elasticity.

- Take $p = 2$, $R(x)$ stays close in $L^2$ to its average on $Q$: Korn-style inequality:

\[
\frac{1}{|Q|} \int_{Q} |Df - (Df)_Q|^2 dx \leq C \sup_{x \in Q} |E(x) - I|^2.
\]

- Another motivation in Moser’s paper to study the Harnack inequality for the solution of a divergence-form elliptic equation:

\[
\sum_{i,j=1}^{n} \partial_i (a_{ij})(x) \partial_j u = 0, \quad a_{ij}(x)'s \text{ are } L^\infty \text{ and uniformly elliptic.}
\]

- Moser gave a third proof of the Hölder regularity of such $u$ after De Giorgi and John Nash. The proof of Harnack’s inequality implies Hölder continuity by an elementary argument.
Another feasible motivation for BMO

Another motivation to define BMO space can be found in the gap of index for Sobolev and Morrey embedding theorems:

- Sobolev embedding: Continuous embeddings of Sobolev spaces into appropriate $L^q$ spaces:
  
  \[ k \in (0, n), \quad 1 < p < \frac{n}{k}, \quad W^{k,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n) \quad \text{if} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{k}{n}; \]

  - If $k \in (0, n)$ is further an integer, the inclusion also exists when $p = 1$.
  - Special case $k = 1$: Gagliardo-Nirenberg-Sobolev inequality.
  - $k = n$: the inclusion holds when $p = \infty$. 

[5/36]
Usefulness of $L^\infty$

The Morrey embedding theorem describes continuous embeddings of Sobolev spaces into appropriate Hölder spaces.

- For $k \in (0, n)$ and $\frac{n}{k} < p < \infty$, the inclusion exists for $\lambda = k - n/p : W^{k,p}(\mathbb{R}^n) \hookrightarrow \Lambda^\gamma(\mathbb{R}^n)$.
- $p = \frac{n}{k}$: Unknown for this case currently.
- As usual, we may also expect to imbed $W^{k,p}$ into $L^\infty$ for $p = n/k$.
- Set $k = 1$ and $p = n$. The embedding into $L^\infty$ only holds when $n = 1$.
- Such claim is not universally right! The only thing we can say is about the embedding into $L^q, p \leq q < \infty$ when $1 \leq p < \infty$.
- Counterexample: $u(x) = \log \left( \log \left( \frac{1}{1+|x|} \right) \right)$. It is in $W^{1,n}(B_1)$ for $n > 1$, but not in $L^\infty(B_1)$.

Another failure of $L^\infty$ can also be seen in the Calderón-Zygmund singular integral operator, which does not preserve $L^\infty$, we omit the details here.
Definition of BMO Space

BMO Space

The bounded mean oscillation space is the vector space of all locally integrable function on $\mathbb{R}^n$ for which

$$\sup_{B} \frac{1}{|B|} \int_{B} |f(x) - f_B| \, dx < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$, and $f_B = \frac{1}{|B|} \int_{B} f$ denotes the mean of $f$ over $B$.

We note that the supremum is 0 for any constant function, so the definition above does not actually induce a proper norm.

To make the definition more make sense, we define BMO as the set of all equivalence functions (modulo equality almost everywhere) of locally integrable functions modulo additive constants. This set carries naturally the structure of a vector space, the above supremum does then define a norm, and BMO is complete under this norm. Thus it is a Banach space.
Small exercise: For a function $u$ whose BMO ”norm” is $K$, we can claim that if $u$ is bounded, then $|u(x) - u(y)| \leq \sqrt{2}K$ in any cube $C_0$.

A simple triangle inequality shows the definition is equivalent to if we replace $f_B$ as any constant $a_B$ depending on $B$, following John-Nirenberg’s original setting.

The function in BMO space keeps scaling invariance and translation invariance.

As usual, we can find the equivalence to the ”norm” in which we replace $B$ by $Q$, arbitrary cube whose sides are parallel to the coordinate axes.
From the study of harmonic analysis, we may find that it is strongly related to the sharp maximal function, which is a variant of Hardy-Littlewood maximal function learnt in real analysis:

$$f^\#(x) = \sup_{B \ni x} \inf_{b \in \mathbb{C}} \frac{1}{|B|} \int_B |f(x) - b| \, dx,$$

where the maximum is taken over all ball balls containing $x$. (The equivalence is verified later)

We thus know that a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is in BMO if and only if $f^\# \in L^\infty$.

If you have learnt something about the property of sharp maximal operator, we will find another motivation to regard BMO as an alternative limit of $L^p$ as $p \to \infty$ since $\|f^\#\|_{L^p}$ is comparable to $\|f\|_{L^p}$ for all $1 < p < 1$, while $\|f\|_{\text{BMO}}$ is comparable to $\|f^\#\|_{\infty}$.

We omit the details on sharp maximal function. We add this section for those interests in harmonic analysis.
The BMO space is closely related to $L^\infty$ in different ways:

- It is easy to find that all bounded functions defined in $\mathbb{R}^n$ are also BMO functions. But the opposite is not right: $\log |x|$ is in BMO($\mathbb{R}$), but $\text{sgn}(x) \log |x|$ is not.
- Furthermore, we have the ”norm”-estimate: $\|f\|_{\text{BMO}} \leq 2\|f\|_{L^\infty}$ by observing that $|f - f_Q|_Q \leq 2|f|_Q \leq 2\|f\|_{L^\infty}$.
- Most transparent fact: the norm for BMO scales and translates in the same way as for $L^\infty$.
- Similarity of value: John-Nirenburg inequality, which further asserts that any BMO function is nearly in $L^\infty$. The result is non-trivial, see Garnett-Jones’ paper for detailed discussion.
- Similarity of function space: We will may the Sobolev embedding theorem for $p = n/k, k \in (0, n)$ by using Poincaré’s inequality only when $k = 1$, while for other cases, we shall use Riesz potential or $H^1 - \text{BMO}$ duality. We omit the proof here.
Proof of equivalence between BMO and sharp maximal function

- For any \( b \), we have

\[
\inf_{b \in \mathbb{C}} \frac{1}{|B|} \int_B |f(x) - b| \, dx \leq \frac{1}{|B|} \int_B |f(x) - f_B| \, dx
\]

\[
\leq \frac{1}{|B|} \int_B |f(x) - b| \, dx + \frac{1}{|B|} \int_B |b - f_B| \, dx
\]

\[
\leq \frac{2}{|B|} \int_B |f(x) - b| \, dx;
\]

- The opposite side is given by taking inf over \( \mathbb{C} \).
- An optimal choice of \( b \) improves upon \( f_B \) by at most a factor of two.
- Note that \( b \) denotes an arbitrary complex number which is permitted to depend on \( B \).
One Motivation for J-N’s inequality

- $f$: real-valued function on $\mathbb{R}^n$ such that $\|f\|_{\text{BMO}} = 1$.
- We have $|f - f_B|_B \leq 1$.
- By Markov’s inequality, we have
  \[ |\{x \in B : |f(x) - f_B| \geq \lambda\}| \leq |B|/\lambda \]
  for any $\lambda > 0$.
- Nontrivial for $\lambda > 1$, $f$ only exceeds its average by say 10 on at most 1/10 of any ball/cube.
- Iterate this fact to give far better estimates as $\lambda \to \infty$.
- Important principle in harmonic analysis: bad behavior on a small exceptional set can often be iterated away if we know that the exceptional sets are small at every scale.
- $f$ only exceeds its average by 20 on at most 1/10 of 1/10 of a given ball (i.e. on at most 1/100 of the ball), $f$ exceeds its average by 30 on at most 0.1%, and so forth, leading to a bound which tails off exponentially in $\lambda$ rather than polynomially.
John-Nirenberg’s inequality

John-Nirenberg (1962)

For function \( f \in \text{BMO}(\mathbb{R}^n) \) with \( \|f\|_{\text{BMO}} \neq 0 \), there exists constants \( C_1, C_2 > 0 \) depending only on dimension \( n \), such that for any cube \( Q \) in \( \mathbb{R}^n \), we have

\[
\left| \{ x \in Q : |f - f_Q| \geq \lambda \} \right| \leq C_1 \exp \left( -C_2 \frac{\lambda}{\|f\|_{\text{BMO}}} \right) |Q|
\]

for any \( \lambda > 0 \).

Here we note that in their original paper, they actually proved the version for \( a_Q \), a number depending on \( Q \) instead of \( f_Q \), however the proof keeps almost the same expect some minor changing. We state the theorem using \( f_Q \) for more common applications.
Sufficient condition for BMO

Before we come the proof of John-Nirenberg’s inequality, we first prove its converse which is much easier to verify inspired from simple calculation. We can also use it to verify that $\log |x|$ is in $\text{BMO}((0,1)^n)$:

**Proposition**

Let $u : Q \to \mathbb{R}$ be a measurable function such that for some $b > 0$, $B \geq 0$ and any cube $C \subset Q$, there exists a constant $a_C \in \mathbb{R}$, such that

$$\left| \{ x \in C : |u - a_C| > \sigma \} \right| \leq B e^{-b\sigma} |C|, \forall \sigma \geq 0,$$

then $u \in \text{BMO}(Q)$.

The proof of this proposition is a simple step of integration,

$$\frac{1}{2} \int_C |u - u_C| \, dx \leq \int_C |u - a_C| \, dx$$

$$= \int_0^\infty | \{ x \in C : |u(x) - a_C| \geq \sigma \} | \, d\sigma \leq \frac{B}{b} |C|.$$
Some consequences

BMO implies locally $L^p$

If $p \in [1, \infty)$ and $u \in \text{BMO}(Q)$, then we have

$$\left( \frac{1}{|Q|} |u - u_Q|^p \right)^{1/p} \leq C_{n,p} \|u\|_{\text{BMO}}.$$

In particular, $u \in L^p_{\text{loc}}(\mathbb{R}^n)$, and the supremum of LHS can approximate $\|u\|_{\text{BMO}}$.

This is a direct result from the integration:

$$\frac{1}{|Q|} \int_Q |u - u_Q|^p dx = p \int_0^\infty |\{ x \in Q : |u(x) - u_Q| > s \}| s^{p-1} ds \leq p \int_0^\infty C_1 e^{-C_2 s/\|u\|_{\text{BMO}}} s^{p-1} ds$$

and change of variables by setting $\lambda := C_2 s/\|u\|_{\text{BMO}}$:

$$\frac{1}{|Q|} \int_Q |u - u_Q|^p dx \leq C_1 p \left( \frac{\|u\|_{\text{BMO}}}{C_2} \right)^p \int_0^\infty \lambda^{p-1} e^{-\lambda} d\lambda = C_1 p C_2^{-p} \Gamma(p) \|u\|_{\text{BMO}}^p.$$
### Exponential Integrability of BMO functions

The exponential integrability of a function is defined over any compact subset $K \subset \mathbb{R}^n$ in the sense that $\int_K \exp(c|g(x)|)dx < \infty$ for any $c < n$. For BMO function, the exponential integrability is guaranteed since

$$\int_K \exp(c|u - u_Q|)dx = c \int_0^\infty e^{ct} |\{x \in Q : |u(x) - u_Q| > t\}| \, dt$$

$$\leq cC_1 \int_0^\infty \exp((c - c_2)t) \, dt = \frac{cC_1}{C_2 - c},$$

where we have assumed $\|u\|_{\text{BMO}} = 1$ and $|Q| = 1$. We thus have claimed:

#### Exponential integrability

For any $c < C_2$ there exists $K(c, C_1, C_2)$ such that

$$\frac{1}{|Q|} \int_Q \exp(c|u - u_Q|/\|u\|_{\text{BMO}})dx \leq K(c, C_1, C_2)$$

for BMO function $u$. 

---

*Note: The above content is a natural reading representation of the document section.*
Now we use the proposition to check the previously unverified result, that \( \log|x| \) is in \( \text{BMO}((0, 1)^n) \).

- Fix a cube \( C \), with \( h \) the length of the side of \( C \).
- Define \( \xi = \max_{x \in C} |x|, \eta = \min_{x \in C} |x|, a_C = \log \xi \). Then \( a_C - u \geq 0 \).
- Removing the trivial part, we assume that \( \xi \geq \eta e^\sigma \). Then we have
  \[
  \xi e^{-\sigma} \geq \eta \geq \xi - \text{diam}(C) \geq \xi - \sqrt{nh},
  \]
  and
  \[
  \xi \leq \frac{\sqrt{nh}}{1 - e^{-\sigma}}.
  \]
- \[
  \frac{1}{h^n} |\{ x \in C : |u - a_C| \geq \sigma \}| \leq \frac{1}{h^n} B \xi e^{-\sigma} \leq \frac{(\sqrt{n})^n \omega_n}{(1 - e^{-\sigma})^n} e^{-n\sigma}.
  \]
- For \( \sigma > 1 \) and \( \sigma \leq 1 \) we have the proposition holds with \( b = n \) and
  \[
  B = \max \{ e^n, (\sqrt{n})^n \omega_n (1 - e^{-1})^{-n} \}.
  \]
- From this, we also know that for \( \xi \in L^1 \), the function \( (\xi * \log)(x) \) is also in \( \text{BMO} \) space.

Let $A_0$ be the smallest constant for BMO function $u$ that satisfy:

$$\sup_Q \frac{1}{|Q|} \left| \{ x \in Q : |u(x) - u_Q| > \lambda \} \right| \leq e^{-\lambda/A_0}.$$  

Then there are constants $C_1, C_2$ depending on $n$ such that for $A > A_0$, we have $u = g + f$, where $\|f\|_{\text{BMO}} \leq C_1 A_0$ and $\|g\|_{L^\infty} \leq C_2 \max(A, \lambda)$.

The converse of this result is almost trivial knowing John-Nirenberg’s inequality.

Not knowing Garnett-Jones’s result, you can also get some intuition that BMO should not be much larger than $L^\infty$. We can also prove that for BMO function $f$, for every $\epsilon > 0$ we have

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1 + |x|)^{n+\epsilon}} \, dx < \infty.$$
Idea Behind the Proof: Discrete Logarithm

- We re-visit the idea for motivating J-N’s inequality:
- Consider two cubes $Q_2 \subset Q_1 \subset \mathbb{R}^n$ with $l(Q_2) = l(Q_1)/2$, then we have

$$|f_{Q_1} - f_{Q_2}| \leq \frac{|Q_1|}{|Q_2|} \cdot \frac{1}{|Q_1|} \int |f(y) - f_{Q_1}| \, dy \leq 2\|f\|_{\text{BMO}}.$$ 

- Iterate this operation to $(n + 1)$ cubes $Q_n \subset Q_{n-1} \subset \cdots \subset Q_0$. We have

$$|f_{Q_n} - f_{Q_0}| \leq n \cdot 2^n \|f\|_{\text{BMO}}.$$ 

- The difference of scaling cause an exponential decay of the ”bounded mean oscillation measure”:

- Magnitudes of the corresponding supports: decrease geometrically in order $2^{-n}$.
  Average: increase arithmetically in order $n$.
- Consider discrete logarithm as a concrete example:

$$g(x) = \sum_{n=0}^{\infty} a_n \chi_{[0,2^{-n}]}(x), \, x \in (0, 1) \text{ with } a_n \geq 1 \text{ for } n \geq 0.$$ 

- $g$ is in BMO iff $a_n = O(1)$ and $g(x) \sim |\log(x)|$ for $0 < x < 1/2$. 

Key ingredient: Calderón-Zygmund decomposition

We use a very useful decomposition in harmonic analysis, invented by Calderón and Zygmund in their study of singular integral operators. Intuitively saying, it says all integrable functions can be decomposed into a good part, where the function is bounded by a small number, and a bad part, where the function can be large, but locally has average value zero; and we have a guarantee that the bad part is supported on a relatively small set:

C-Z decomposition

Given a function $f$ which is integrable and non-negative, and given a positive number $\lambda$, there exists a sequence $\{Q_j\}$ of disjoint dyadic cubes such that

1. (good part) $f(x) \leq \lambda$ for almost every $x \notin \bigcup_j Q_j$;
2. (cube requirement) $|\bigcup_j Q_j| \leq \frac{1}{\lambda} \|f\|_1$;
3. (bad part) $\lambda < f_{Q_j} \leq 2^n \lambda$.

The proof of this can be seen in any standard reference on harmonic analysis. We omit it here.
Proof of J-N’s inequality

• WLOG, we may assume that \( \|f\|_{\text{BMO}} = 1 \), so we have

\[
\frac{1}{|Q|} \int_{Q} |f(x) - f_{Q}| dx \leq 1.
\]

• Form the C-Z decomposition of \((f - f_{Q})\) at height 2, which gives a family of disjoint cubes \(\{Q_{1,j}\}\) such that

\[
|f - f_{Q}| \leq 2
\]

for \(x \notin \bigcup j Q_{1,j}\) and

\[
2 < \frac{1}{|Q_{1,j}|} \int_{Q_{1,j}} |f(x) - f_{Q}| dx \leq 2^{n+1}.
\]

• In particular, we have

\[
\sum_{j} |Q_{1,j}| \leq \frac{1}{2} \int_{Q} |f(x) - f_{Q}| dx \leq \frac{1}{2} |Q|,
\]

and

\[
|f_{Q_{1,j}} - f_{Q}| = \left| \frac{1}{|Q_{1,j}|} \int_{Q_{1,j}} (f(x) - f_{Q}) dx \right| \leq 2^{n+1}.
\]
Proof of J-N’s inequality

- Form the C-Z decomposition of \((f - f_{Q_1,j})\) at height 2 on each cube \(Q_{1,j}\), (again we may assume that the \((f - f_{Q_1,j}) \leq 1\))

- We obtain a family of cubes \(\{Q_{1,j,k}\}\) satisfies the following:
  1. \(|f(x) - f_{Q_1,j}| \leq 2\) for \(x \in Q_{1,j} \setminus (\bigcup_k Q_{1,j,k})\);
  2. \(\sum_k |Q_{1,j,k}| \leq \frac{1}{2} |Q_{1,j}|\);
  3. \(|f_{Q_1,j,k} - f_{Q_1,j}| \leq 2^{n+1}\).

- Gather the cubes \(Q_{1,j,k}\) corresponding to all \(Q_{1,j}\)’s and collectively called \(\{Q_{2,j}\}\).

- Now, we have

\[
\sum_j |Q_{2,j}| \leq \frac{1}{4} |Q|
\]

and for \(x \notin \bigcup_j Q_{2,j}\),

\[
|f(x) - f_Q| \leq |f(x) - f_{Q_1,j}| + |f_{Q_1,j} - f_Q| \leq 2 + 2^{n+1} \leq 2 \cdot 2^{n+1}.
\]
Proof of J-N’s inequality

- Repeat the process infinitely:
- For each $N$, we have a family of disjoint cubes $\{Q_{N,j}\}$, such that

$$ |f(x) - f_Q| \leq N \cdot 2^{N+1} $$

and

$$ \sum_{j} |Q_{N,j}| \leq \frac{1}{2^N} |Q| $$

for $x \notin \bigcup_j Q_{N,j}$. 
Proof of J-N’s inequality

- Fix $\lambda \geq 2^{n+1}$, and choose $N$ such that $N2^{n+1} \leq \lambda < (N + 1)^{n+1}$.

  $$\left| \{ x \in Q : |f(x) - f_Q| \geq \lambda \} \right| \leq \sum_{j} |Q_{N,j}| \leq \frac{1}{2^N} |Q|$$

  $$= e^{-N \log 2} |Q| \leq e^{-C_2 \lambda} |Q|.$$  

  (pick $C_2 = \log 2/2^{n+2}$)

- If $\lambda < 2^{n+1}$, then $C_2 \lambda < \log \sqrt{2}$. Hence,

  $$|\{ x \in Q : |f(x) - f_Q| \geq \lambda \}| \leq |Q| \leq \exp(\log \sqrt{2} - C_2 \lambda) |Q|$$

  $$= \sqrt{2} \exp(-C_2 \lambda) |Q|$$

  so we can take $C_1 = \sqrt{2}$. 

Here we should note that the constants in these estimates are not optimal, for example, we can change the height for C-Z decomposition to derive improvements for previous result. The optimal constant of $C_2 = 2/e$ is achieved by using rearrangement argument which we will not state here. For $C_1$, the problem remains open while the main direction using rearrangement function keeps the same, where we can take $C_1 = e^{4/e}/2$. See [Ana] for more discussion on such approach.

There are many alternative ways to prove John-Nirenberg’s inequality, for example, we can reduce this problem to a direct consequence of BMO – $H^1$ duality by using the property for $H^1$ function and Riesz’s representation theorem, see Stein’s harmonic analysis for the proof.
BMO and $L^p$

First we note that the John-Nirenburg’s inequality can be extended considering the $L^p$ norms, so that the case of BMO maps corresponds to the limit as $p \to \infty$:

**Corollary**

For any $p \in [1, \infty)$ and $u \in L^p(Q)$ we define

$$K_p^p(u) := \sup \left\{ \sum_i |Q_i| \left( \frac{1}{|Q_i|} |u(x) - u_{Q_i}| \right)^p, Q = \bigcup_i Q_i \right\}.$$ 

Then there exists a constant $c = c(p, n)$ such that

$$\|u - u_Q\|_{L^w_p} \leq c(p, n)K_p(u),$$

where $\| \cdot \|_{L^w_p}$ refers to the weak $L^p$ norm.

The proof is basically the same as John-Nirenberg’s inequality, the goal being to prove the polynomial decay:

$$|\{x \in Q : |u(x) - u_Q| > t\}| \leq \frac{c(p, n)}{t^p} K_p(u), t > 0$$

instead of an exponential decay.
We still invoke iteration on C-Z decomposition of \( u - u_{Q_j} \) at level \( 2^j \mu \) where \( \mu \) is given according to \( \mu \geq 1/|Q|^{1/p} \).

After \( k^{th} \) step we invoke the C-Z decomposition for the function \((u - u_{Q_{k-1}})1_{Q_{k-1}}\) at level \( 2^{2(k-1)} \mu \) and we get a collection of open, disjoint sub-cubes \( \{Q_{k,j}\} \) of \( Q_{k-1} \) so that \( |u(x)| \leq 2^{2k-1} \mu \) a.e. in \( Q_{k-1} \setminus \bigcup_j Q_{k,j} \) and

\[
\sum_{\text{all } j's} |Q_{k,j}| \leq \frac{\left(\int_Q |u(x)| \, dx\right)^{(k-1)/p'}}{\phi(\mu, k)}
\tag{1}
\]

where

\[
\phi(u, k) = (2^{2(k-1)} \mu)(2^{2(k-2)} \mu)^{1/p'} (2^{2(k-3)} \mu)^{1/p'^2} \cdots \mu^{1/p'^{k-1}}.
\]

The proof reduces to the estimate for \( \phi \) by property of series and taking \( \mu = 2/|Q|^{1/p} \). We omit the details here.
BMO and $L^p$

- As a substitute for $L^\infty$, we would like to be able to use it in interpolation theory. This is indeed possible, as the following theorem shows:

**Stanpacchia’s interpolation**

Let $T$ be a linear operator bounded on $L^{p_0}(\mathbb{R}^n)$ for some $1 \leq p_0 < \infty$ and bounded from $L^\infty \to \text{BMO}$. Then $T$ is bounded on $L^p(\mathbb{R}^n)$ for any $p_0 < p < \infty$.

- The proof also comes from Calderón-Zygmund decomposition and comparison of Hardy-Littlewood maximal function and sharp maximal function. We omit the proof here.

- After knowing BMO – $H^1$ duality, we can derive an analog for $H^1$ space by using Marcinkiewicz’s interpolation theorem.
An Unexpected Application

An important problem in classical analysis is to find smoothness condition for the existence of a continuous derivative. Weiss and Zygmund proved a statement of giving some conditions:

**Weiss-Zygmund smoothness condition**

Let $f$ be $2\pi$-periodic and continuous. Suppose for some $\beta > 1/2$, that $f(x + t) + f(x - t) - 2f(x) = O\left(t/|\log t|^\beta\right)$ uniformly in $x$ as $t \to 0^+$. Then $f$ is absolutely continuous and its derivative $f'$ is in $L^p[0, 2\pi]$ for every $p > 1$.

- The original proof of this result is fairly short but utilizes a number of deep results including a theorem of Littlewood and Paley.
- The John-Nirenberg inequality approaches this result more directly, more general and as well shows that the derivative $f'$ is not merely in each $L^p$ space but also of BMO.
A New Proof

The proof relies on the following result:

**Lemma**

Let $u \in L^1(C_0)$, where $C_0$ is a finite cube in $\mathbb{R}^n$. Assume that there is a constant $K$ and a constant $\beta > 1/2$ such that if $C_1$ and $C_2$ are any two equal subcubes having a full $(n - 1)$-dimensional face in common, then $|u_{C_1} - u_{C_2}| \leq K/(1 + |\log h|^\beta)$, where $h$ is the common side length.

Then $u$ is a BMO function.

The proof of using this lemma is direct:

- WLOG, assume $F$ is a smooth function, then we only need to estimate $\|F'\|_{L^p}$.
- For $n = 1$, we have the quantity as in the lemma for $F'$. $\equiv f$.
- The proof is complete by giving the John-Nirenberg inequality for $u$. 
Sketch of Proof

- Sub-divide each edge of a sub-cube $C$ into $2^N$ equal parts to get $2^{nN}$ equal cubes $\{C_r\}_{1 \leq r \leq 2^{nN}}$.

- \[
\frac{1}{|C|} \int_C |u - u_C| \, dx = \lim_{N \to \infty} 2^{-nN} \sum_r |u_r - u_C|.
\]

- The proof reduces to claim $2^{-nN} \sum_r |u_r - u_C| \leq \kappa(\beta, K, n)$.

- By Cauchy-Schwartz’s inequality,
  \[
  2^{-nN} \sum_r |u_r - u_C| \leq \left[2^{-nN} \sum_r |u_r - u_C|^2\right]^{1/2} := a_N^{1/2}.
  \]

- The proof reduces to show $a_N$’s are uniformly bounded so that
  \[
  a_{N+1} \leq a_N + \left(\frac{nK}{1+|\log h|\beta}\right)^2, \text{ where } h = K/2^{N+1}.
  \]

- Since $a_0 = 0$, $a_{N+1} \leq n^2 K^2 \sum_{j=1}^{\infty} \left(1 + j \log 2 - \log k\right)^{-2} \leq \kappa^2$ for some constant $\chi$ independent of $K$. The convergence holds since $\beta^{1/2}$.

- The remaining proof relies on the property of series and subdivision of subcubes to derive iteration results.
Before we end the presentation, let us introduce some other applications of John-Nirenberg inequality not discussed in the presentation:

- Back to the initial question on nonlinear elastic boundary value problem, John provides an attractive answer in 1972 by proving the uniqueness of nonlinear elastic equilibria when the boundary displacement is fixed and only deformations with uniformly small strain are considered. This is done using John-Nirenberg’s inequality.

- The John-Nirenberg’s inequality acts an important rule in the study for the supersolution of elliptic PDE of divergence form, deriving generally weak Harnack’s inequality by Moser, which resulting in the Hölder regularity of the solution. The proof is greatly simplified by using J-N inequality. Some generalization can further be considered. See Moser’s paper or [Han] for a systematic discussion.

- If we know something about harmonic analysis, by $T(1)$ theorem there are some singular integral operator related to BMO which can be extended to $L^2$. John-Nirenberg’s inequality acts an important rule in the proof of $T(1)$ theorem.
Duality for BMO Space

- Similar to the failure of preservance of $L^\infty$ space for singular integral operator, a similar problem still happens for $L^1$. (Hilbert transform of an integrable function is not in general integrable)
- We define a subspace of $L^1$ whose image under any singular integral is in $L^1$: $H^1$ space:

**Space Atomic $H^1$**

The $H^1$ space is defined by the set of linear combination of special atoms $a_j$, with coefficients $\lambda_j$ are complex and have finite sum of countable addition.

The atom $a'_j$s are defined as complex-valued functions defined on $\mathbb{R}^n$ which is supported on a cube $Q$ and is such that $\int_Q a = 0$ and $\|a\|_\infty \leq 1/|Q|$. 
The norm of $H^1$ is defined by

$$\|f\|_{H^1} = \inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j a_j \right\}.$$

So the $H^1$ is a Banach space.

We can also check that singular integrals are bounded from $H^1$ to $L^1$. Just similar to the case for $L^\infty$ to BMO.

A natural thinking is to connect $H^1$ and BMO since they are the substitutes of $L^1$ and $L^\infty$ respectively and they have similar properties as the dual to each other:

Fefferman-Stein, 1971

There exists constants $0 < c \leq C < \infty$ such that

$$c \|f\|_{\text{BMO}} \leq \sup_{g : \|g\|_{H^1} \leq 1} \left| \int f(x)g(x)dx \right| \leq C \|f\|_{\text{BMO}}$$

and every bounded linear functional on $H^1$ is of the above type.
Useful References

- Anatolii Korenovskii. Mean Oscillations and Equimeasurable Rearrangements of Functions.
- Javier Duoandikoetxea. Fourier Analysis.
Useful References

- Brian Thomson. Symmetric Properties of Real Functions.